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That function of x whose first coefficient is unity and whose roots are the doubled cosines of all the primitive divisions of the circle in respect to k contains no prime factors except such as are divisors of, or else when increased or diminished by unity, are divisible by k. This may be called again the *Exclusional or Negative Theorem of Twin Factors;* and on the other hand the more extraordinary theorem which asserts (on evidence not yet conclusive) that the function of x above defined, when made homogeneous in x, y, will represent (at all events for the case of $k = 9$) every prime number of the form $ki \pm 1$, or else certain specific multiples of any such number, may be called the *Inclusional or Representational Theorem of Twin Factors.*

A New Proof of the Theorem of Reciprocity.

BY DR. JULIUS PETERSEN, *of Copenhagen, Denmark.*

LET a and b be two odd prime numbers, $a < b$; form the equation

$$(2n+1)\,a - 2mb = r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for all the odd numbers $2n+1$ up to $p-2$, and choose m so that r, which will be termed a remainder, shall lie between $-b$ and $+b$. The absolute numerical values of these remainders will be all different, and therefore they must be the odd numbers $1, 3, \ldots, (b-2)$, of which, however, some may be negative. According as the number of these negative values is even or odd we will write

$$\left(\frac{a}{b}\right) = +1 \quad \text{or} \quad \left(\frac{a}{b}\right) = -1 .$$

From among the equations (1) we will take out those in which the remainders lie between $-a$ and $+a$; there is one such equation for every value of $2m$ up to $a-1$; these equations may be written under the form

$$(a-2m)\,b - (b-2n-1)\,a = r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which are evidently the equations for the determination of the sign of $\left(\frac{b}{a}\right)$. Therefore $\left(\frac{a}{b}\right)$ and $\left(\frac{b}{a}\right)$ will have the same or opposite signs, according as the number of the remainders between $-b$ and $-a$ is even or odd. For such remainders

$$-b < (2n+1)\,a - 2mb < -a,$$

or, putting $m = n - k$, $a = b - 2\alpha$,

$$2n + 2 > \frac{(k+1)\,b}{a} > 2n + 1\,.$$

Therefore there is a negative remainder between $-a$ and $-b$ for every term in the series

$$\frac{b}{a}\,,\quad \frac{2b}{a}\,,\quad \frac{3b}{a}\,,\,\ldots\ldots,\quad \frac{(a-1)\,b}{a}\cdot$$

in which the greatest contained integer is odd; now any two terms in the above series at equal distances from its extremities have the sum b, and therefore the integer parts of such two (their sum being $b - 1$) are both even or both odd; for α odd the number of terms is even, and the number of negative remainders between $-a$ and $-b$ therefore even; for α even the number depends upon the mean term $\frac{1}{2}\,b$; it is even if $b = 4i + 1$, odd if $b = 4i + 3$; hence we shall always have

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2}\cdot\frac{b-1}{2}}\,.$$

By multiplying the equations (1) by c we obtain

$$\left(\frac{ac}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{c}{b}\right);$$

and, remarking that, b being a prime number, the multiplication* of the equations (1) shows that $\left(\frac{a}{b}\right)$ is in accord with Legendre's notation, we have the proof of the extended (erweitert) theorem.

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* Viz: multiplying together the equations (1) in the form

$$\begin{array}{rl}
1\,.\,a \equiv r_1 & (\text{mod. } 2b)\,, \\
3\,.\,a \equiv r_2 & (\quad \text{''} \quad)\,, \\
5\,.\,a \equiv r_3 & (\quad \text{''} \quad)\,, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
(b-2)\,.\,a \equiv r_{\frac{b-1}{2}} & (\quad \text{''} \quad)\,,
\end{array}$$

and noticing that $r_1\,.\,r_2\,.\,r_3\,\ldots\,r_{\frac{b-1}{2}} = \left(\frac{a}{b}\right) 1\,.\,3\,.\,5\,\ldots\,(b-2)$, we have

$$a^{\frac{b-1}{2}} \equiv \left(\frac{a}{b}\right) (\text{mod. } 2b)\,,$$

and thus $\left(\frac{a}{b}\right) = +1$ or -1 is the criterion that a shall be a quadratic residue or non-residue of b.

EDS.